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Monotonicity and Mutability*

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1. Introduction

This is a study of systems which possess several steady states and whose solutions approach the set of the steady states as time tends to infinity. One is especially interested in the property of mutability, which implies the ability of the system to execute transitions from one steady state to another. It will be shown that this type of behavior is secured if the system possesses some very simple properties of monotonicity.

The general concept of mutability includes many cases studied before. Closely related to this paper are especially J. Moser's work on nonoscillating networks [1] and G. A. Leonov's paper [2], concerned with global asymptotic stability. Mutable systems have been often encountered, more or less explicitly, in investigating the limits of the methods of the theory of stability (see, for instance, J. A. Nohel and D. F. Shea [3]).

In this paper one studies mutable systems of special structures, aiming at identifying systems with high mobility, whose transitions from state to state are sharp, fast and easily controllable. The method of study is based on a comparison approach (section 4) which establishes similarities between the investigated system and a "model". A simple example is the system

$$\xi' + \xi + h_2(\eta) = 0 \tag{1.1}$$

$$\eta' + \eta + h_1(\xi) = 0 \tag{1.2}$$

where

$$h_1(\rho) = h_2(\rho) = \begin{cases} (2+\delta)\,\rho & \text{if } \quad |\rho| < 1 \\ 2 + \delta\rho & \text{if } \quad \rho \geqslant 1 \\ -2 + \delta\rho & \text{if } \quad \rho \leqslant -1 \end{cases} \tag{1.3}$$

and where δ is a number in the interval $(0, 1)$. There are three steady states: $(0, 0)$ (a saddle point), $(-2/(1-\delta), 2/(1-\delta))$, (a stable node) and $(2/(1-\delta), -2/(1-\delta))$ (another stable node). The phase plane method shows that every

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orbit will approach one of these steady states and suggests procedures for triggering transitions between these steady states (e.g., by translating h_1 or h_2).

This example illustrates two general features of the subject: First, a mutable system is necessarily nonlinear. Second, mutability is, essentially, a global property.

If this example is modified by introducing some delays, or more complicated nonlinear functions, or differential equations of higher orders, then a simple analysis as the one above is no longer possible. Consider, for instance, "convolution systems" of the form

$$\xi + g_2 * h_2(\eta) + f_1 = 0 \tag{1.4}$$

$$\eta + g_1 * h_1(\xi) + f_2 = 0 \tag{1.5}$$

where the f_i, g_i and h_i, for $i = 1, 2$, are given functions (suppose, for definiteness, that the h_i are uniformly Lipschitzian, the f_i are integrable and uniformly continuous, the g_i are bounded and the functions $t \mapsto tg_i(t)$ are integrable, for $i = 1, 2$). One asks whether every *bounded* solution will approach the set of the constant solutions of the system. The answer is affirmative (Section 7) provided some conditions of monotonicity are satisfied: the h_i are increasing (in a strict sense, as defined in Section 3) and the g_i are decreasing (one only has to eliminate the singular case in which the absolutely continuous part of g_1 or g_2 vanishes; it will suffice that, on some intervals, g_1 and g_2 have strictly negative derivatives). Under various supplementary conditions on the h_i, one obtains other properties: some estimates for the solutions and the conclusion that every solution is bounded.

This is not the most general result established, but the fact that one can obtain so simple and easily applicable criteria for a potentially complicated problem is one of the main motivations of this paper.

This relation between mutability and monotonicity parallels the well known connection between convexity and stability (see J. J. Levin [4], A. Halanay [5], J. A. Nohel and D. F. Shea [3]).

In order to secure clear-cut transitions, it is desirable that all the characteristic values of the linearized system (about an arbitrary stationary solution) be real. (The mutable system with this property will be called "hyperbolic".) It turns out that this additional property is satisfied if (in addition to the conditions mentioned before) g_1 and g_2 are "completely monotonic" in the sense used, e.g., in the theory of Laplace–Stieltjes integrals (see D. V. Widder [6]).

In obtaining these results, one has implicitly used some methods, introduced recently by the author [7, 8, 9]. Other results, on related problems, were obtained by J. J. Levin and D. F. Shea [10, 11, 12]. See also P. Hartman [13, Ch. XIV].

A list of the principal symbols, concepts and results of the paper is given in Section 2, intended to serve as a guide and a reference while reading the paper.

2. Symbols and Reference Information

The symbols R, R^n, R_+ (or $[0, \infty)$) and $|\cdot|$ have their usual meanings. The symbol $L^1(R_+)$ is abbreviated as L^1. The same is true for C_0, L^2 and L^∞, which otherwise have their usual meanings (e.g., as in W. Rudin [14]). One uses the abbreviations $|\cdot|_1$, $|\cdot|_2$ and $|\cdot|_\infty$ instead of the more usual notations $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$. All these symbols are used for scalar valued functions as well as for vector-valued or matrix-valued functions (the specific case is clear from the context). If $f \in C_0 \cap L^1$ (which obviously implies that $f \in L^2 \cap L^\infty$) one uses the notation $|f| = |f|_1 + |f|_\infty$.

The scalar product is denoted simply by $(\cdot, \cdot)$, since there is no real danger of confusion with intervals in R^1 or points in R^2. The convolution-multiplication is denoted by $*$. The Fourier transforms—distinguished by a hat—are applied only to functions defined on R_+ and one uses the formula $\hat{x}(\omega) = \int_0^\infty e^{-i\omega t} x(t)\, dt$.

Symbols Implying Information about Domains and Ranges

The letter y always denotes a vector in R^n. One also writes $y = (y_i)$, where the y_i are the components of y and it is understood that $i = 1, 2, \ldots, n$.

The letters x, z, f, ϕ and also $\tilde{x}$, x_T, $\tilde{f}$ etc., denote functions from R_+ into R^n. The letters h, $\tilde{h}$ and h^ρ denote diagonal functions from R^n into R^n, i.e. functions of the form $h(y) = (h_i(y_i))$, where the h_i are functions from R into R; g denotes a function from R_+ into $R_+^{n\times n}$.

By $h(x)$ one denotes a function from R_+ into R^n, defined as $(h(x))(t) = h(x(t))$. The same is true for $h(z)$. However, in agreement with the above conventions, $h(y)$ is a vector in R^n.

Fixed Symbols

P is a nonsingular, symmetric $n \times n$ matrix; I is the identity matrix.

By L and ϵ one denotes two numbers—fixed in all the proofs—which satisfy the conditions $L > 1 > \epsilon > 0$ and $1 - n\epsilon\, |P| > 0$. By k and K (with or without indices) one denotes constants. In Section 3, these constants depend only on P, L and ϵ. In Sections 4–7 these constants depend only on P, L, ϵ and g.

General assumptions on h: h is uniformly Lipschitzian and strictly increasing (with constants L and ϵ, as explained in Section 3); moreover $h(0) = 0$ (by performing a suitable translation, this condition can always be satisfied if the equation $y + Ph(y) = 0$ has at least a solution).

Principal Equations

(S) $\quad x + Pg * h(x) + f = 0, f \in C_0 \cap L^1$: The "convolution equation" (Section 4)

(M) $\quad z' + z + Ph(z) + \phi = 0, \phi \in C_0 \cap L^2, z(0) = 0$: The "model" (Section 3)

(St) $y + Ph(y) = 0$: equation of the stationary solutions (Section 3)

(SM) $x + Pg * h(x) + f = 0, \quad f \in C_0 \cap L^1$ the "comparison system"
$z' + z + Ph(x) = 0, \quad z(0) = 0$ (Section 5)

Principal Concepts

"Mutable" and "strictly mutable": (a) for models (Def. 3.1) (b) for convolution systems (Def. 4.1).

"Regularly decreasing" (Def. 6.1).

"Hyperbolic mutability" (Def. 8.2).

Main Results

"Monotonicity implies mutability" Prop. 3.2 and 3.3 and Th. 7.2.

"Complete monotonicity implies hyperbolic mutability" Th. 8.3.

3. Models

An "n-dimensional model" has the form

$$\text{(M)} \qquad z' + z + Ph(z) + \phi = 0, \quad \phi \in C_0 \cap L^2, \quad z(0) = 0, \quad t \geqslant 0.$$

where P is a symmetric, nonsingular $n \times n$ matrix and h is a uniformly Lipschitzian function from R^n into R^n (therefore (M) has a unique continuous solution z on R_+).

To the model (M) one associates the equation

$$\text{(St)} \qquad y + Ph(y) = 0, \qquad y \in R^n$$

whose solutions are called "stationary solutions". Graphical interpretations are often informative about the solutions of (St). One assumes that the set of the stationary solutions is bounded and one introduces the notation

$$\sigma(h) = \sup\{| y | : y + Ph(y) = 0\}. \tag{3.1}$$

It is assumed that h is "diagonal", i.e. $h(y) = (h_i(y_i))$ where $h_i: R \to R$, $i = 1,..., n$. (The methods of this paper work however for larger classes of functions, at the price of some sacrifice of simplicity and briefness.) One assumes that "h is uniformly Lipschitzian and strictly increasing, with constants L and ϵ", i.e. that $| h(y) - h(\tilde{y})| \leqslant L | y - \tilde{y} |$, for every $y, \tilde{y} \in R^n$ and that the functions $\xi \mapsto h_i(\xi) - \epsilon\xi$, $i = 1,..., n$, are increasing. The matrix P and the numbers L and ϵ will be fixed in all the proofs. One assumes that $L > 1 > \epsilon > 0$ and $1 - n\epsilon | P | > 0$.

To simplify some estimates, one assumes that $h(0) = 0$. One can show that the systems studied are translation invariant (see Section 4 below) and the above condition amounts to assuming that the equation $y + Ph(y) = 0$ has at least a solution $y \in R^n$.

If $h(0) = 0$ and if h is uniformly Lipschitzian and strictly increasing with constants L and ϵ, one will say, for short, that h satisfies the "general assumptions".

In this section, the constants denoted by k or K (with or without indices) depend only on P, L and ϵ.

3.1. Definition. A. The model (M) is said to be mutable if there exists $K > 0$ such that every bounded solution z of (M) approaches the set of the stationary solutions as $t \to \infty$ (i.e. $\lim_{t\to\infty} \inf\{| z(t) - y | : y + Ph(y) = 0\} = 0$) and satisfies the inequality $| z |_\infty \leqslant K(| \phi |_2 + \sigma(h))$.

B. (M) is said to be strictly mutable if it is mutable and all its solutions are bounded.

One first obtains a partial property of mutability:

3.2. Proposition. *If h satisfies the general assumptions, then every bounded solution of* (M) *approaches the set of the stationary solutions as t tends to infinity.*

Proof. The assumptions on h imply that h is invertible and the inverse h^{-1} is again diagonal, uniformly Lipschitzian and strictly increasing, with constants $1/\epsilon$ and $1/L$. The same is true for the function h^c, defined as $h^c(y) = -h^{-1}(-y)$. Define W_i and W by $W_i(\xi) = \int_0^\xi h_i^c(\rho)\, d\rho$ and $W(y) = \sum_{i=1}^n W_i(y_i)$. Let z be a solution of (M) and let T be an arbitrary positive number. Then

$$\int_0^T (P^{-1}z', h^c(P^{-1}z))\, dt = W(P^{-1}z(T)) - W(P^{-1}(z(0)). \tag{3.2}$$

Define J as $J(T) = -\int_0^T (P^{-1}z', z)\, dt$. Since P^{-1} is symmetric,

$$J(T) = -\tfrac{1}{2}(P^{-1}z(T), z(T)) + \tfrac{1}{2}(P^{-1}z(0), z(0)). \tag{3.3}$$

On the other hand, (M) implies that $z = -h^c(P^{-1}(z' + z + \phi))$. Therefore one can write $J(T)$ as

$$J(T) = J_1(T) + J_2(T) + J_3(T) \tag{3.4}$$

where

$$\begin{aligned} J_i(T) &= \int_0^T (P^{-1}z', v_i)\, dt, \qquad i = 1, 2, 3, \\ v_1 &= h^c(P^{-1}z) \\ v_2 &= h^c(P^{-1}(z + \phi)) - h^c(P^{-1}z) \\ v_3 &= h^c(P^{-1}(z' + z + \phi)) - h^c(P^{-1}(z + \phi)). \end{aligned}$$

Then $J_1(T)$ is precisely the left hand member of (3.2). Use CBS and the fact that h^c is uniformly Lipschitzian to find k_1 such that

$$| J_2(T)| \leqslant k_1 | \phi |_2 \left(\int_0^T | z' |^2 \, dt\right)^{1/2}.$$

Notice that $(h^c(y) - h^c(\tilde{y}), y - \tilde{y}) \geqslant 1/L \, | y - \tilde{y} |^2$ and find $k_2 > 0$ such that

$$J_3(T) \geqslant k_2 \int_0^T | z' |^2 \, dt.$$

Now (3.3) and (3.4) imply that

$$k_2 \int_0^T | z' |^2 \, dt - k_1 | \phi |_2 \left(\int_0^T | z' |^2 \, dt\right)^{1/2}$$
$$\leqslant -W(P^{-1}z(T)) + W(P^{-1}z(0)) + \tfrac{1}{2}(P^{-1}z(0), z(0)) - \tfrac{1}{2}(P^{-1}z(T), z(T)), \tag{3.5}$$

which is the key result of the proof. If z is bounded, then, after completing the square in (3.5), one finds that $z' \in L^2$.

One uses now (M) repeatedly to accumulate more and more information about z. First, (M) implies that z' is bounded and therefore z is uniformly continuous. Then (M) shows that z' is uniformly continuous; one concludes that $z' \to 0$ as $t \to \infty$. Finally, since $\phi \to 0$, one finds, from (M), that $z + Ph(z) \to 0$ as $t \to \infty$. Since h is continuous, the conclusion follows.

For further needs, notice the following consequence of the proof: There exists $k > 0$ such that, if the solution z of (M) is bounded, then $z' \in L^2$ and

$$| z' |_2 \leqslant k(| \phi |_2 + \sigma(h)). \tag{3.6}$$

Indeed, under the general assumptions on h, one can find k_3 such that $| W(y)| \leqslant k_3 | y |^2$. Then the conclusion follows from (3.5), with $z(0) = 0$ and $T \to \infty$, since now one knows that z approaches the set of the stationary solutions as $t \to \infty$.

To secure mutability, one needs supplementary assumptions on h. The definitions imply that, if $| y | > \sigma(h)$, then $y + Ph(y) \neq 0$. It will be enough to strengthen somewhat this condition:

3.3. PROPOSITION. *In addition to the general assumptions on h, suppose also that if $| y | > L\sigma(h)$ then $| y + Ph(y)| > \epsilon \, | y |$. Then* (M) *is mutable.*

Proof. By Prop. 3.2 and by (3.6), it suffices to prove that there exist two constants K_1 and K_2 such that, if z is bounded, then $| z |_\infty \leqslant 2K_1 | z' |_2 + K_2 | \phi |_2 + L\sigma(h)$. One claims that this inequality is certainly satisfied for

$$K_1 = 2/\epsilon^{1/2}; \qquad K_2 = 2/(\epsilon K_1). \tag{3.7}$$

Suppose the contrary and find $t_1 > 0$ such that

$$| z(t_1)| > 2K_1 | z' |_2 + K_2 | \phi |_2 + L\sigma(h). \tag{3.8}$$

Consider the interval $J = [t_1 , t_1 + K_1^2]$. By CBS,

$$| z(t) - z(t_1)| \leqslant \int_{t_1}^{t} | z'(\tau)| \, d\tau \leqslant (t - t_1)^{1/2} | z' |_2 .$$

Therefore

$$| z(t)| \geqslant | z(t_1)| - | z(t) - z(t_1)| \geqslant | z(t_1)| - K_1 | z' |_2 , \qquad \text{in } J. \tag{3.9}$$

A first consequence of (3.8) and (3.9) is that $| z(t)| > L\sigma(h)$ in J. Therefore (M) gives $| z'(t)| = | z(t) + Ph(z(t)) + \phi(t)| > \epsilon | z(t)| - | \phi(t)|$, or

$$2 | z'(t)|^2 > \epsilon^2 | z(t)|^2 - 2 | \phi(t)|^2. \tag{3.10}$$

Another consequence of (3.8), (3.9) is that $| z(t)| > K_1 | z' |_2 + K_2 | \phi |_2$, or $| z(t)|^2 > K_1^2 | z' |_2^2 + K_2^2 | \phi |_2^2$. Therefore (3.10) implies that $2 | z'(t)|^2 > \epsilon^2 K_1^2 | z' |_2^2 + \epsilon^2 K_2^2 | \phi |_2^2 - 2 | \phi(t)|^2$ in J. Integrating on J one finds that

$$2 \int_J | z'(t)|^2 \, dt > \epsilon^2 K_1^4 | z' |_2^2 + \epsilon^2 K_1^2 K_2^2 | \phi |_2^2 - 2 \int_J | \phi(t)|^2 \, dt.$$

The left hand member is $\leqslant 2 | z' |^2$ whereas the right hand member (with K_1 and K_2 from (3.7)) is $\geqslant 2 | z' |_2$. The contradiction proves the proposition.

The conditions on h need to be further strengthened in order to secure strict mutability. Given a function h and a number $\rho > 0$, define h^ρ as $h^\rho(y) = (h_i^\rho(y_i))$, where

$$h_i^\rho(\xi) = \begin{cases} h_i(\xi) & \text{if } \ | \xi | < \rho \\ h_i(\rho) + \epsilon(\xi - \rho) & \text{if } \ \xi \geqslant \rho \\ h_i(-\rho) + \epsilon(\xi + \rho) & \text{if } \ \xi \leqslant -\rho \end{cases} \tag{3.11}$$

If h satisfies the general assumptions, so does h^ρ, with the same constants L and ϵ. It will be convenient to write $h^\infty = h$.

3.4. Proposition. *In addition to the general assumptions on h, suppose that* $\sup_{\rho \in [0,\infty]} \sigma(h^\rho) < \infty$ *and that, for every* $\rho \in [0, \infty]$, *if* $| y | > L\sigma(h^\rho)$ *then* $| y + Ph^\rho(y)| > \epsilon | y |$. *Then* (M) *is strictly mutable.*

Proof. One has only to show that all the solutions of (M) are bounded. Suppose the contrary and choose ϕ and $t_1 > 0$ such that the solution z of (M) satisfies the inequality

$$| z(t_1)| > K(| \phi |_2 + \sup_{\rho \in [0,\infty]} \sigma(h^\rho)) \tag{3.12}$$

Take $\rho = \sup_{t\in[0,t_1]} |z(t)|$ and let $\tilde{z}$ be the solution of (M) with h replaced by h^ρ. By the definition of h^ρ and the uniqueness of solutions,

$$\tilde{z}(t_1) = z(t_1). \tag{3.13}$$

On the other hand, the choice of ϵ and the definition of h^ρ imply that $\tilde{z}$ is bounded. Therefore, by Prop. 3.3, $|\tilde{z}|_\infty \leq K(|\phi|_2 + \sigma(h^\rho))$. From (3.12) and (3.13) it then follows that $|\tilde{z}(t_1)| > |\tilde{z}|_\infty$ and one obtains a contradiction.

As an example, consider the system (1.1)–(1.2) with $h_1(\rho) = h_2(\rho) = 2\rho$. Then it is easy to see that (1.1)–(1.2) is mutable but not strictly mutable (since there are unbounded solutions). In this case, h satisfies the conditions of Prop. 3.3 but does *not* satisfy the conditions of Prop. 3.4.

4. Convolutions Systems

The convolution systems studied in this paper have the form

(S) $$x + Pg * h(x) + f = 0, \qquad f \in C_0 \cap L^1,$$

where

(i) P is a symmetric and nonsingular $n \times n$ matrix (as in Section 3),

(ii) h satisfies the general assumptions of Section 3 (with the same constants L and ϵ),

(iii) g is a diagonal and bounded function, from R_+ into $R_+^{n\times n}$, the function $t \mapsto tg(t)$ is in L^1 and

$$\int_0^\infty g(t)\,dt = I. \tag{4.1}$$

Notice that, except in singular cases, (4.1) is just a scaling condition: it can be satisfied if one replaces g by gH and h by $H^{-1}h$, where H is a diagonal, positive definite, $n \times n$ matrix, suitably chosen.

Under the above assumptions, (S) has a unique continuous solution x, from R_+ into R^n (see, e.g., R. K. Miller [15, Theorems 9.1 and 9.2, Ch. I, Th. 6.1, Ch. II and p. 127]).

In the rest of this paper, the letters k and K, with or without indices, denote constants which depend only on P, L, ϵ and g.

The condition on $tg(t)$ in (iii) above implies a property of translation-invariance (considered essential for the mutability of the system): Let y_0 be an arbitrary vector in R^n and define $\tilde{x}$, $\tilde{h}$ and $\tilde{f}$ by $\tilde{x} = x - y_0$, $\tilde{h}(y) = h(y + y_0) + P^{-1}y_0$ and $\tilde{f}(t) = f(t) + P\int_t^\infty g(\tau)\,d\tau\,P^{-1}y_0$. Then one obtains the relation $\tilde{x} + Pg * \tilde{h}(\tilde{x}) + \tilde{f} = 0$, similar to (S). Moreover, if the function $t \mapsto tg(t)$ is in L^1, then $\tilde{f}$ belongs to L^1 (as one can see using Fubini's theorem) and therefore $\tilde{f}$ satisfies the requirements in (S).

Because of this property of translation-invariance, the condition $h(0) = 0$, adopted in the general assumptions on h, can always be satisfied if $y + Ph(y) = 0$ has a solution.

The conditions in (iii) play also a (technical) role in the proof of Lemma 6.2, below.

One will freely use the basic properties of convolution (e.g., as in E. Hewitt and K. Stromberg [16, Theorems (21.31)–(21.33)]).

As in Section 3, a vector y from R^n is said to be a stationary solution of (S) iff $y + Ph(y) = 0$ (thus (S) and (M) have the same stationary solutions). As before, the set of the stationary solutions is assumed to be bounded and one denotes by $\sigma(h)$ the upper bound, as in (3.1).

The definitions of mutability have to be slightly adjusted. To simplify the statements, if $f \in C_0 \cap L^1$ one writes $|f| = |f|_1 + |f|_\infty$.

4.1. Definition. A. (S) is said to be mutable if there exists K such that every bounded solution x of (S) approaches the set of the stationary solutions as $t \to \infty$ and satisfies the inequality $|x|_\infty \leqslant K(|f| + \sigma(h))$.

B. (S) is said to be strictly mutable if it is mutable and all its solutions are bounded.

Precise conditions of mutability will be given in Section 7. For further needs, notice the following:

4.2. *Remark.* If h has the form $h(x) = \epsilon x + h_0(x)$ and if h_0 is bounded, then the corresponding continuous solution x of (S) is bounded.

Indeed, then x satisfies the equation $x + \epsilon Pg * x = \tilde{f}$, where $\tilde{f} = -f - Pg * h_0(x)$. Since $\tilde{f}$ is obviously bounded and since $1 - n\epsilon\, |P| > 0$ (by the choice of ϵ) the result is easily established, using (4.1).

5. Comparison Systems

One compares a system (S), as in Section 4, with a model (M), as in Section 3, to find conditions under which the mutability of (M) implies the mutability of (S). The "comparison system" has the form

$$\begin{aligned} x + Pg * h(x) + f &= 0, \qquad f \in C_0 \cap L^1 \\ z' + z + Ph(x) &= 0, \qquad z(0) = 0. \end{aligned} \tag{SM}$$

5.1. Proposition. *Suppose that, for every bounded solution* (x, z) *of* (SM), $x - z \in L^2$. *Suppose also that every bounded solution of* (M) *approaches the set* *of the stationary solutions as* $t \to \infty$. *Then every bounded solution of* (S) *has the same property.*

Proof. Let x be a bounded solution of (S). Then the solution z of (SM) is bounded and, by assumptions, $x - z \in L^2$. From (SM) it follows that z satisfies (M) with

$$\phi = P(h(x) - h(z)). \tag{5.1}$$

If one proves that

$$\phi \in C_0 \cap L^2 \tag{5.2}$$

then the assumption on (M) implies that z approaches the set of the stationary solutions of (M), which is identical with the set of the stationary solutions of (S). Therefore, to finish the proof, it suffices to show that $x - z \to 0$ and that ϕ satisfies (5.2).

From (S), since $f \in C_0 \cap L^1$, $g \in L^1$ and $h(x) \in L^\infty$, it easily follows that x is uniformly continuous; and from (SM) one sees that z is bounded and uniformly continuous. Thus $x - z$ is uniformly continuous; one concludes that $x - z \to 0$ as $t \to \infty$. The obtained conclusions also imply that $\phi \in C_0$. Finally, since h is uniformly Lipschitzian, one finds (using (5.1)) that $\phi \in L^2$. Thus (5.2) is established and the proof is complete.

5.2. PROPOSITION. *Suppose that there exists K' such that, if (x, z) is a bounded solution of* (SM), *then* $x - z \in L^2$ *and*

$$| x - z |_2 \leqslant K'(| f | + \sigma(h) + (| f | \, | x |_\infty)^{1/2}). \tag{5.3}$$

Suppose also that (M) *is mutable* (*Def.* 3.1.A). *Then* (S) *is mutable* (*Def.* 4.1.A).

Proof. Use the general assumptions on h to find that $h(x) - h(z) \in L^2$ and that there exists k_1 with the property

$$| h(x) - h(z)|_2 \leqslant k_1 \, | x - z |_2 \,. \tag{5.4}$$

Since (M) is mutable, it follows (from (5.1)) that

$$| z |_\infty \leqslant \tilde{K}(| x - z |_2 + \sigma(h)). \tag{5.5}$$

Now (S) can be rewritten as $x = -Pg * h(z) - Pg * (h(x) - h(z)) - f$ and hence $| x |_\infty \leqslant | P | \, | g |_1 \, | h(z)|_\infty + | P | \, | g |_2 \, | h(x) - h(z)|_2 + | f |_\infty$. Use (5.3)–(5.5) and the general assumptions on h to find k_2 such that

$$| x |_\infty \leqslant k_2(| f | + \sigma(h) + (| f | \, | x |_\infty)^{1/2}).$$

Complete the square to obtain the conclusion.

5.3. PROPOSITION. *Suppose that the assumption* (5.3) *of Proposition* 5.2 *is satisfied for every h which satisfies the general assumptions of Section* 3 *and the conditions of Proposition* 3.4. *Then* (S) *is strictly mutable* (*Def.* 4.1.B).

Proof. One only has to prove that all the solutions of (S) are bounded. The arguments are essentially the same as in the proof of Propisition 3.4 (now the boundedness of the solutions of (S)—with h replaced by h^ρ—follows from Remark 4.2).

6. MONOTONIC SYSTEMS

One considers the convolution system (S) assuming that h is strictly increasing (as explained in Section 3) and that g is decreasing, in the following sense:

6.1. DEFINITION. A function g—satisfying the assumptions in (iii) Section 4 —is said to be regularly decreasing if every diagonal element of g is decreasing, right continuous and its absolutely continuous part does not vanish identically.

The role of the regularly decreasing functions in the property of mutability will be shown in Section 7. Here one lists some of the properties which will be used in further proofs.

6.2. LEMMA. *Suppose that g is regularly decreasing. Let q be a diagonal element of g. Let* μ *be the unique Borel measure (cf., e.g., W. Rudin* [14, *Theorem* 8.14]) *concentrated on* $(0, \infty)$ *and associated to q according to the formula* $q(t) = \mu((t, \infty))$, *for every* $t \geqslant 0$. *Let* $\hat{q}$ *be the Fourier transform of q. Then the following statements are true*:

A. $\hat{q}(0) = 1$ *and*

$$\hat{q}(\omega) = \frac{1}{i\omega}\Big(\mu(R_+) - \int_{R_+} e^{-i\omega t}\, d\mu(t)\Big), \qquad \omega \neq 0 \tag{6.1}$$

B. *There exists* $k_0 > 1$ *such that, for every real* ω,

$$\left| \hat{q}(\omega) - \frac{1}{1 + i\omega} \right|^2 \leqslant k_0\omega^2/(1 + \omega^2)^2 \tag{6.2}$$

$$\operatorname{Re}(i\omega\hat{q}(\omega)) \geqslant k_0\omega^2/(1 + \omega^2) \tag{6.3}$$

$$| \hat{q}(\omega)|^2 \geqslant k_0/(1 + \omega^2). \tag{6.4}$$

C. *Let u be a function in* $L^1 \cap L^\infty$, *from* R_+ *into R. Define the function r by*

$$r(t) = u(t)\, \mu(R_+) - \int_{[0,t]} u(t - \tau)\, d\mu(\tau), \qquad t \geqslant 0. \tag{6.5}$$

Define also the function v, from R_+ into R, by $v = q * u$. Then r is in $L^1 \cap L^\infty$, v is in $C_0 \cap L^1$, v is absolutely continuous on R_+ and

$$v' = r, \qquad a.e.\ in\ R_+. \tag{6.6}$$

Moreover, denoting by $\widehat{v'}$ (resp. by $\hat{v}$) the Fourier transforms of v' (resp. v) one has

$$\widehat{v'}(\omega) = i\omega\hat{v}(\omega), \qquad for\ every\ real\ \omega. \tag{6.7}$$

D. *In addition, if ξ is a function in L^1, from R_+ into R, and if one defines w by*

$$w(\tau) = \int_{R_+} (\xi(t) - \xi(t+\tau))\, u(t)\, dt, \qquad \tau \geqslant 0 \tag{6.8}$$

then

$$\int_0^\infty v'\xi\, dt = \int_{R_+} w\, d\mu. \tag{6.9}$$

These results are relatively straightforward consequences of Fubini's theorem and other basic properties of the integral. It will suffice to mention briefly some details.

From the assumptions it follows that $q \geqslant 0$ and that the function $t \mapsto tq(t)$ is in L^1. Then Fubini's theorem implies that $\int_{R_+} t^2\, d\mu(t) < \infty$. This condition is used without further mention (together with the other assumptions of integrability for the functions involved) to justify various operations of reversing the order of integration or passing to limits under the integral sign.

To obtain (6.1), consider $\hat{q}(\omega) = \int_0^\infty e^{-i\omega t} q(t)\, dt$, replace $q(t)$ by $\int_{(t,\infty)} d\mu$ and use Fubini's theorem. To prove (6.2), it suffices to show that $\sup_{\omega\in R} | i\omega\hat{q}(\omega)| < \infty$ (which follows from (6.1)) and that

$$\sup_{\omega>0} |(\hat{q}(\omega) - 1/(1+i\omega))/i\omega\,| < \infty. \tag{6.10}$$

Observe first that

$$\sup_{\omega>0} \left| \int_0^\infty q(t) \sin(\omega t)/\omega\, dt \right| \leqslant \int_0^\infty tq(t)\, dt < \infty. \tag{6.11}$$

Next, since q is decreasing and $t \mapsto tq(t) \in L^1$, one can find k such that $q(t) \leqslant k/t^2$, for every $t > 0$. Denote $q(t)(1 - \cos(\omega t))/\omega$ by E and observe that, for every $\omega > 0$, one has $\int_0^{\pi/\omega} E\, dt \in [0, k\pi]$ and $\int_{\pi/\omega}^\infty E\, dt \in [0, 2k/\pi]$; therefore $\sup_{\omega>0} \int_0^\infty E\, dt < \infty$. This and (6.11) imply (6.10). To prove (6.3), it suffices (by the continuity of $\hat{q}$) to show that (a) $\mathrm{Re}(i\omega\hat{q}(\omega)) > 0$ for every $\omega \neq 0$, (b) $\underline{\lim}_{\omega\to\infty} \mathrm{Re}(i\omega\hat{q}(\omega)) > 0$, and (c) $\lim_{\omega\to 0+} \mathrm{Re}(i\hat{q}(\omega)/\omega) > 0$. These properties follow from (6.1) (here one needs the assumption about the absolutely continuous part of q; cf. J. A. Nohel and D. F. Shea [3, Section 4]).

As for (6.4), use (6.3), $\hat{q}(0) = 1$ and the continuity of $\hat{q}$.

Since $u \in L^1 \cap L^\infty$, one sees, from (6.5) that $r \in L^1 \cap L^\infty$. The integral $\int_0^T \int_{[0,t]} u(t-\tau)\, d\mu(\tau)\, dt$ $(T \geqslant 0)$ can be brought to the form $\int_0^T \int_{[0,T-t]} d\mu(\tau)\, u(t)\, dt$ or $\int_0^T (\mu(R_+) - q(T-t))\, u(t)\, dt$. This proves that $v(T) = \int_0^T r(t)\, dt$ for every $T \geqslant 0$ and therefore v is absolutely continuous and satisfies (6.6). To obtain the conclusion that $v \in C_0 \cap L^1$, notice that $q \in L^1 \cap L^2$ and use the general properties of the convolution $v = q * u$.

To prove (6.7), substitute $v(t) = \int_0^t v'(\tau)\, d\tau$ in $\hat{v}(\omega) = \lim_{T\to\infty} \int_0^T e^{-i\omega j} v(t)\, dt$, apply Fubini's theorem and use the fact that $v \in C_0 \cap L^1$. (The case $\omega = 0$ is examined directly.)

For (6.9), use (6.5)–(6.6) to write the left hand member of (6.9) as $I_1 - I_2$, where $I_1 = \int_0^\infty u(t)\, \xi(t)\, dt\, \mu(R_+)$ and $I_2 = \int_0^\infty \int_{[0,t]} u(t-\tau)\, d\mu(\tau)\, \xi(t)\, dt$. Fubini's theorem can be applied to I_2, bringing it to the form $\int_{R_+} \int_\tau^\infty u(t-\tau)\, \xi(t)\, dt\, d\mu(\tau)$, and hence $I_2 = \int_{R_+} \int_{R_+} u(t)\, \xi(t+\tau)\, dt\, d\mu(\tau)$. On the other hand, since $\mu(R_+) = \int_{R_+} d\mu$, one can obviously write $I_1 = \int_{R_+} \int_{R_+} u(t)\, \xi(t)\, dt\, d\mu(\tau)$. This gives (6.9).

7. Mutable Systems

Now one can prove one of the main results of the paper:

7.1. Theorem. *Suppose that h satisfies the general assumptions of Section 3 and that g is regularly decreasing (Definition 6.1). Then every bounded solution of (S) approaches the set of the stationary solutions as t tends to infinity.*

Proof. Introduce the comparison system (SM). By Proposition 5.1 and Proposition 3.2, it suffices to show that, if x is bounded, then the solution (x, z) of (SM) has the property that $x - z \in L^2$. To write this condition in a more convenient form, let λ and ν be the solutions of the equations

$$\lambda + \theta P g * \lambda + f = 0 \tag{7.1}$$

$$\nu' + \nu + \theta P \lambda = 0, \qquad \nu(0) = 0 \tag{7.2}$$

where $\theta = \epsilon/2$. Observe that (since $1 - n\theta \mid P \mid > 0$, since g satisfies (4.1) and since $f \in C_0 \cap L^1$) λ and ν are in $C_0 \cap L^1$ (and hence also in L^2 and L^∞). Moreover, there exists k_1 such that

$$\max(\mid \lambda \mid_1, \mid \lambda \mid_\infty, \mid \lambda \mid_2) \leqslant k_1 \mid f \mid \tag{7.3}$$

$$\max(\mid \nu \mid_1, \mid \nu \mid_\infty, \mid \nu \mid_2) \leqslant k_1 \mid f \mid \tag{7.4}$$

Let now x be a bounded solution of (S) and let z be given by (SM). Let T be a positive number and define u_T, x_T and z_T by

$$u_T(t) = \begin{cases} -h(x(t)) + \theta x(t) & \text{if} \quad t \in [0, T] \\ 0 & \text{if} \quad t > T \end{cases} \tag{7.5}$$

$$x_T = Pg * u_T - \theta P g * x_T \tag{7.6}$$

$$z_T' + z_T = P u_T - \theta P x_T, \qquad z_T(0) = 0. \tag{7.7}$$

From (7.5)–(7.7) it follows that $x_T \in L^2$ and $z_T \in L^2$. Moreover, by uniqueness,

$$x_T(t) = x(t) - \lambda(t) \qquad \text{if} \quad t \in [0, T] \tag{7.8}$$

$$z_T(t) = z(t) - \nu(t) \qquad \text{if} \quad t \in [0, T]. \tag{7.9}$$

Notice also that (by (7.3), (7.4), (7.8) and (7.9)) there exists k_2 such that

$$| x - z |_2 \leqslant \varliminf_{T \to \infty} | x_T - z_T |_2 + k_2 | f | \tag{7.10}$$

Therefore it suffices to prove that

$$\sup_{T \geqslant 0} | x_T - z_T |_2 < \infty. \tag{7.11}$$

The proof of (7.11) is in several steps:

I. Introduce the Fourier transform $\hat{g}$ and define v_T and J by

$$v_T = g * u_T \tag{7.12}$$

$$J(T) = \int_0^\infty (v'_T, x_T)\, dt = (1/2\pi) \int_{-\infty}^\infty (i\omega \hat{v}_T, \hat{x}_T)\, d\omega. \tag{7.13}$$

The last equality in (7.13) is obtained by using (6.7) for each term of the scalar product and by applying Parseval's identity. Notice that these operations (and the similar ones which follow) are legitimate, because the functions involved belong to L^1 and to L^2. Here and in the following one uses abbreviations of the form $\int_{-\infty}^\infty (i\omega \hat{v}_T, \hat{x}_T)\, d\omega$ instead of $\int_{-\infty}^\infty (i\omega \hat{v}_T(\omega), \hat{x}_T(\omega))\, d\omega$. From (7.6) one sees that $\hat{v}_T = \theta \hat{g} \hat{x}_T + P^{-1} \hat{x}_T$ and hence

$$J(T) = (1/2\pi) \int_{-\infty}^\infty (i\omega(\theta \hat{g} \hat{x}_T + P^{-1} \hat{x}_T), \hat{x}_T)\, d\omega.$$

Since P^{-1} is symmetric, the term containing P^{-1} vanishes. Using (6.3) for every component of g one finds k_3 such that

$$J(T) \geqslant k_3 \int_{-\infty}^\infty (\omega^2/(1 + \omega^2))\, | \hat{x}_T |^2\, d\omega.$$

From (7.6) one finds that $\hat{x}_T = P\hat{g}(\hat{u}_T - \theta \hat{x}_T)$ and therefore, using (6.4) for every diagonal element of g, one finds k_4 such that

$$J(T) \geqslant k_4 \int_{-\infty}^\infty (\omega^2/(1 + \omega^2)^2)\, | \hat{u}_T - \theta \hat{x}_T |^2\, d\omega. \tag{7.14}$$

On the other hand, from (7.6) and (7.7) it follows that $\hat{x}_T - \hat{z}_T =$

$P(\hat{g} - I/(1 + i\omega))(\hat{u}_T - \theta\hat{x}_T)$. Therefore, using (6.2) componentwise, one finds k_5 such that

$$\int_{-\infty}^{\infty} | \hat{x}_T - \hat{z}_T |^2 \, d\omega \leqslant k_5 \int_{-\infty}^{\infty} (\omega^2/(1 + \omega^2)^2) \, | \hat{u}_T - \theta\hat{x}_T |^2 \, d\omega. \tag{7.15}$$

Use (7.14), (7.15) and Parseval's identity to find k_6 such that

$$| x_T - z_T |_2^2 \leqslant k_6 J(T). \tag{7.16}$$

Now all one has to do to finish the proof is to show that $\sup_{T \geqslant 0} J(T) < \infty$.

II. Since x is bounded, one sees, from (7.5), that u_T is bounded. Applying (6.5) and (6.6) componentwise, one concludes that v_T' is also bounded and that there exists k_7 such that

$$| v_T' |_\infty \leqslant k_7 \, | x |_\infty \tag{7.17}$$

Now rewrite J as

$$J(T) = J_1(T) + J_2(T) \tag{7.18}$$

where $J_1(T) = -\int_0^\infty (v_T', \lambda) \, dt$ and $J_2(T) = \int_0^\infty (v_T', x_T + \lambda) \, dt$. Use (7.3), (7.17) and the expression of $J_1(T)$ to find k_8 such that

$$| J_1(T) | \leqslant k_8 \, | x |_\infty \, | f |. \tag{7.19}$$

To finish the proof, it suffices now to show that $\sup_{T \geqslant 0} J_2(T) < \infty$.

III. Define $\tilde{x}_T$, $\tilde{x}_i$ and $\tilde{h}_i$ by

$$\tilde{x}_T = x_T + \lambda, \qquad \tilde{x}_T = (\tilde{x}_i) \tag{7.20}$$

$$\tilde{h}_i(\xi) = h_i(\xi) - \theta\xi. \tag{7.21}$$

Use (7.5), (6.8)–(6.9) (with the μ_i defined as in Lemma 6.2, for the corresponding elements of g) to rewrite J_2 as

$$J_2(T) = \sum_{i=1}^{n} \int_{R_+} w_i \, d\mu_i \tag{7.22}$$

where

$$w_i(\tau) = -\int_0^T (\tilde{x}_i(t) - \tilde{x}_i(t + \tau)) \, \tilde{h}_i(x_i(t)) \, dt \tag{7.23}$$

From (7.8) and (7.20) it follows that

$$\tilde{x}_i(t) = x_i(t) \qquad \text{if} \quad t \in [0, T].$$

Therefore (7.23) can be rewritten as

$$w_i(\tau) = -\int_0^T (\tilde{x}_i(t) - \tilde{x}_i(t+\tau))\, \tilde{h}_i(\tilde{x}_i(t))\, dt. \tag{7.24}$$

Since x is bounded and x_T is given by (7.5)–(7.6)—and since λ is also bounded and satisfies (7.3)—one can find K_1 such that

$$|\, \tilde{x}_i \,|_\infty \leqslant K_1(|\, x\,|_\infty + |\, f\,|). \tag{7.25}$$

Define now the functions V_i, from R into R, by

$$V_i(\xi) = \int_0^\xi \tilde{h}_i(\rho)\, d\rho. \tag{7.26}$$

Since the $\tilde{h}_i$ (7.21) are increasing and uniformly Lipschitzian, one easily obtains the inequality

$$-(\tilde{x}_i(t) - \tilde{x}_i(t+\tau))\, \tilde{h}_i(\tilde{x}_i(t)) \leqslant V_i(\tilde{x}_i(t+\tau)) - V_i(\tilde{x}_i(t)).$$

Now (7.24) implies that

$$w_i(\tau) \leqslant \int_0^T V_i(\tilde{x}_i(t+\tau))\, dt - \int_0^T V_i(\tilde{x}_i(t))\, dt,$$

and hence

$$w_i(\tau) \leqslant \int_T^{T+\tau} V_i(\tilde{x}_i(t))\, dt - \int_0^\tau V_i(\tilde{x}_i(t))\, dt, \qquad \tau \geqslant 0. \tag{7.27}$$

From (7.21), (7.26), $\theta = \epsilon/2$ and the general assumptions on h, it follows that $V_i \geqslant 0$. Therefore (7.27) becomes

$$w_i(\tau) \leqslant \int_T^{T+\tau} V_i(\tilde{x}_i(t))\, dt. \tag{7.28}$$

Moreover, by (7.25), (7.26) and the properties of h, one can find K_2 such that $|\, V_i(\tilde{x}_i(\tau))| \leqslant K_2(|\, x\,|_\infty + |\, f\,|)^2$. Therefore (7.28) gives

$$w_i(\tau) \leqslant 2K_2\tau(|\, x\,|_\infty + |\, f\,|)^2. \tag{7.29}$$

Since the measures μ_i are associated to g as explained in Lemma 6.2 and since $g \in L^1$, one easily sees, by Fubini's theorem, that $\int_{R_+} t\, d\mu_i(t) < \infty$. With this information, one uses now (7.29) and (7.22) to find K_3 such that $J_2(T) \leqslant K_3(|\, x\,|_\infty + |\, f\,|)^2$. This shows that $\sup_{T\geqslant 0} J_2(T) < \infty$ and ends the proof.

7.2. THEOREM. A. *Under the same assumptions an in Theorem* 7.1, *suppose also that h satisfies the assumptions of Proposition* 3.3. *Then* (S) *is mutable.*

B. *Suppose, in addition, that h satisfies the conditions of Proposition* 3.4. *Then* (S) *is strictly mutable.*

Proof. Let x be a bounded solution of (S) and let z be the corresponding solution of (SM). To prove the first part of the theorem it is enough to show that $x - z$ satisfies (5.3).

Observe that since, by Theorem 7.1, x approaches the set of the stationary solutions of the system, one has

$$\overline{\lim_{t\to\infty}} \, |x(t)| \leqslant \sigma(h). \tag{7.30}$$

By (7.10), (7.16), (7.18) and (7.19), it suffices to find K_4 such that $\overline{\lim}_{T\to\infty} J_2(T) \leqslant K_4(\sigma(h) + |f|)^2$. From (7.22), (7.28) and the argument presented above following (7.29), one sees that it suffices to find K_5 with the property

$$\overline{\lim_{T\to\infty}} \int_T^{T+\tau} V_i(\tilde{x}_i(t))\, dt \leqslant K_5\tau(\sigma(h) + |f|)^2. \tag{7.31}$$

One can further see (from (7.20), (7.21), (7.3) and the general assumptions on h) that this will certainly take place if there exists K_6 such that

$$\overline{\lim_{T\to\infty}} \sup_{t\geqslant T} |x_T(t)| \leqslant K_6\sigma(h). \tag{7.32}$$

To obtain (7.32) notice that (by using (7.30), (7.5), (7.21) and the general assumptions on h) one can find K_7 and K_8 such that, if $t \geqslant K_7$, then, for every $T \geqslant 0$,

$$|u_T(t)| \leqslant K_8\sigma(h). \tag{7.33}$$

If $T \geqslant K_7$, one can write uniquely u_T as $u_T = u_T{}^1 + u_T{}^2$, where $u_T{}^1(t) = 0$ for $t > K_7$ and $u_T{}^2(t) = 0$ for $t \leqslant K_7$. Then (7.33) can be written as

$$|u_T{}^2|_\infty \leqslant K_8\sigma(h). \tag{7.34}$$

Now x_T can be written as $x_T = x_T{}^1 + x_T{}^2$ where, in agreement with (7.6), the $x_T{}^j$ satisfy the equations

$$x_T{}^j = Pg * u_T{}^j - Pg * x_T{}^j, \qquad j = 1, 2.$$

Use (7.34) to find K_9 such that, for every $T \geqslant 0$,

$$|x_T{}^2|_\infty \leqslant K_9\sigma(h). \tag{7.35}$$

In order to prove (7.32) it suffices now to show that

$$\overline{\lim_{T\to\infty}} \sup_{t\geqslant T} |x_T{}^1(t)| = 0. \tag{7.36}$$

But since u_T^1 vanishes outside $[0, K_7]$, the functions $g * u_T^1$ and x_T^1 do not depend on T, as long as $T > K_7$. Moreover, one can easily see that $x_T^1 \in C_0$. This gives (7.36) and concludes the proof of the first part of the theorem.

The obtained conclusions, together with Proposition 5.3, prove also immediately the second part of the theorem.

8. Hyperbolic Systems

Given a stationary solution of (S), one considers the corresponding linearized equation. It has the form

$$x + Pg * Hx + f = 0 \tag{L}$$

where H is a diagonal matrix whose all diagonal elements are strictly positive. It is desirable to introduce conditions which make it impossible for the linearized system to have nonreal eigenvalues (the presence of complex eigenvalues would tend to make the transitions from state to state more sluggish). The following definition is confined to a particular case, sufficient for the present purposes.

8.1. Definition. Let g be as in Section 4. For $\operatorname{Re} s > 0$ consider the analytic function

$$G(s) = \int_0^\infty e^{-st} g(t)\, dt \tag{8.1}$$

(Laplace transform). Suppose that there exists an analytic function (denoted again by G) defined everywhere, except possibly on the real negative semi-axis $s \leqslant 0$, and equal to (8.1) for $\operatorname{Re} s > 0$. Let s be a complex number, not on the semi-axis $s \leqslant 0$. Under these assumptions, one says that s is an eigenvalue of (L) if

$$\det(I + PG(s)\, H) = 0. \tag{8.2}$$

This definition, when applicable, agrees with the standard ones, in the particular cases in which (L) corresponds to an ordinary differential equation or to a delay-differential equation (see, e.g., J. Hale [17]).

8.2. Definition. A strictly mutable system (S), for which the assumptions in Definition 8.1 are satisfied, is said to be hyperbolically mutable if, for every stationary solution y of (S), h is differentiable at y and the linearized system $x + Pg * h'(y)\, x + f = 0$ does not have any nonreal eigenvalues.

For the needs of the next theorem, recall that a function χ, from R_+ into R, is said to be completely monotonic if (1) it is indefinitely differentiable on $(0, \infty)$,

(2) $(-1)^k \chi^{(k)} \geqslant 0$, for $k = 0, 1, \ldots$, on $(0, \infty)$ and (3) $\chi(0+) < \infty$. Recall also that every completely monotonic function on R_+ can be represented in the (Laplace–Stieltjes) form

$$\chi(t) = \int_0^\infty e^{-\alpha t}\, dq(\alpha), \tag{8.3}$$

for some bounded and increasing (i.e., nondecreasing) function q, from R_+ into R_+ ("Bernstein's theorem"; see, for instance, D. V. Widder [6, p. 154]).

The diagonal matrix g will be said to be completely monotonic if every diagonal element of g is completely monotonic.

8.3. THEOREM. *In addition to the conditions of Theorem* 7.2 *part* B, *suppose also that h is differentiable and g is completely monotonic on R_+. Then* (S) *is hyperbolically mutable.*

Proof. Using Bernstein's representation (8.3) componentwise, one can write

$$g(t) = \int_0^\infty e^{-\alpha t}\, dQ(\alpha),$$

where Q is diagonal and all its diagonal elements are bounded, increasing functions. (Implicitly, Q is assumed to satisfy other conditions, because of the general assumptions on g in (iii) Section 4.) For $\operatorname{Re} s > 0$, the Laplace transform of g can be brought (by Fubini's theorem) to the form

$$G(s) = \int_0^\infty (1/(s+\alpha))\, dQ(\alpha). \tag{8.4}$$

The same expression defines an analytic function for every complex s, except possibly the values of s on the negative semi-axis $s \leqslant 0$. (See, e.g., J. Dieudonné [18, (13.8.6)(iii)].)

Suppose that (S) is not hyperbolically mutable and let s and H be such that $\operatorname{Im} s \neq 0$, $\det(I + PG(s)H) = 0$, H is diagonal and all its diagonal elements are strictly positive. Find $y \neq 0$ such that $y + PG(s)Hy = 0$. Then $G(s)Hy = -P^{-1}y$. Taking the scalar product with y gives

$$(y, G(s)Hy) = -(y, P^{-1}y). \tag{8.5}$$

Since P^{-1} is symmetric, the right hand member is real. However from (8.4) it follows that

$$\operatorname{Im} G(s) = -\operatorname{Im} s \int_0^\infty (1/|s+\alpha|^2)\, dQ(\alpha),$$

and therefore the diagonal elements of $\operatorname{Im} G(s)H$ are different from zero and of the same sign. Therefore the left hand member of (8.5) is not real—a contradiction.

Remark. A similar proof shows that, if g satisfies the conditions of Theorem 7.2 part B then all the eigenvalues of (L) in the right hand half plane $\operatorname{Re} s > 0$ are real. However in this case (L) may have nonreal eigenvalues in the left hand half plane.

9. An Extremal System

If the example (1.4)–(1.5) from the introduction is reconsidered under the assumptions of Theorem 8.3, it becomes a hyperbolically mutable system. One will prove that this system is even "extremal," in the sense that, if the conditions of monotonicity on h and g are relaxed (even in an "arbitrarily small"-sense) then the property of hyperbolic mutability is no longer satisfied.

Suppose first that all the conditions of Theorem 8.3 are kept unchanged, except that one relaxes the condition on h; namely one replaces the condition of strict monotonicity of h by the requirement that the function $x \mapsto h(x) + \delta x$ be increasing, for some small $\delta > 0$. Then the conclusion of the theorem ceases to hold, as the following example shows:

$$\xi' + \xi + \eta = 0$$
$$\eta' + \eta - \delta\xi = 0.$$

Indeed, one easily sees that the characteristic equation of this system has nonreal roots.

Now one keeps again all the conditions of Theorem 8.3 unchanged, except that one relaxes the conditions on g: Instead of the complete monotonicity of g, one requires the function $t \mapsto g(t) + I\delta e^{-\gamma t}$ to be completely monotonic, for some small $\delta > 0$ and $\gamma > 0$. Then again the conclusion of the theorem does not hold.

To prove this, let γ be a small, strictly positive number and define ω and β by $\omega = \gamma/(1 - 2\gamma^2)^{1/2}$ and

$$\beta = 1/((1/(1 + \omega^2) - 2\gamma^3/(\gamma^2 + \omega^2)).$$

Choose $\gamma > 0$ small enough, such that ω is real and $\beta > 0$. Consider the system

$$x_1' + x_1 + h_2(\eta) = 0, \qquad x_3' + x_3 + h_1(\xi) = 0$$
$$x_2' + \gamma x_2 + 2\gamma^2 h_2(\eta) = 0, \qquad x_4' + \gamma x_4 + 2\gamma^2 h_1(\xi) = 0,$$
$$\xi = x_1 - x_2\,, \qquad \eta = x_3 - x_4\,,$$

which is of the form (1.4)–(1.5) with $g_1(t) = g_2(t) = e^{-t} - 2\gamma^2 e^{-\gamma t}$. Take

$h_1(\rho) = h_2(\rho) = \beta\rho$. Then it is easy to check that the obtained linear system has the following complex solution

$$\xi(t) = -\eta(t) = e^{i\omega t}$$
$$x_1(t) = -x_3(t) = \beta e^{i\omega t}/(1 + i\omega)$$
$$x_2(t) = -x_4(t) = 2\beta\gamma^2 e^{i\omega t}/(\gamma + i\omega).$$

The real part of this solution is a nonconstant, real periodic solution of the system and therefore the system is not even mutable.

There are other examples which show that the mutable systems established in this paper are sometimes extremal. In these cases one has reached natural boundaries of the problem. As a whole, however, the subject of mutability, far from being exhausted, gives rise to many intriguing research questions.

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